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FLAT G-BUNDLES WITH CANONICAL METRICS

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1. Introduction

A great deal of attention has recently been focused on the relationship between the invariant theory of semisimple algebraic group actions on complex algebraic varieties and the behavior of the moment map and quantum data provided by an associated symplectic structure. In particular, it has been shown that the moment map has zero as a regular value precisely when there are stable points, in the sense of geometric invariant theory [13]. This paper discusses an infinite dimensional instance of this correspondence involving flat bundles over compact Riemannian manifolds.

In finite dimensions, this philosophy received its simplest, and earliest, exposition in [11]. There, Kempf and Ness considered the case of a representation of a semisimple algebraic group G over $\mathbf{C}$ on a complex vector space V with some positive definite Hermitian form. Geometric invariant theory picks out a class of G-orbits called the stable ones. To be specific, $v \in V$ is stable if its orbit is closed in V and has the maximum possible dimension. Kempf and Ness observed that Gv is closed in V if and only if it contains a shortest vector. On the other side of the ledger, there is a symplectic structure on V associated with the chosen Hermitian form, and one has the action of the compact subgroup of G which preserves this form. There is a moment map associated with this action, and it turns out that the vanishing of this map at v is equivalent to v being the shortest vector in its orbit under G.

This relationship between symplectic geometry and algebraic geometry has been rephrased in the more sophisticated framework of geometric quantization in, for example, [9]. In another direction, Atiyah and Bott encountered an infinite dimensional instance of this correspondence in their study [1] of the Yang-Mills equations over Riemann surfaces. In that situation, the Kähler manifold in question was the space of all Hermitian connections on a Hermitian vector bundle over a compact Riemann surface. The analogue of G was the gauge group of vector bundle automorphisms. Atiyah and Bott

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noted that the curvature could be interpreted as a moment map, and that a theorem of Narasimhan and Seshadri gave the desired relationship between the moment map and the algebro-geometric notion of a stable vector bundle over an algebraic curve. The results which one would expect for Kähler manifolds of higher dimension have been given by Donaldson [5], in the case of algebraic surfaces, and Uhlenbeck and Yau [19] in general.

We discuss a similar problem related to flat principal G-bundles over compact Riemannian manifolds, where G is a complex semisimple algebraic group. In this situation, we call a flat bundle stable if the image of its holonomy homomorphism is not contained in a proper parabolic subgroup of G. The vanishing of the moment map is, roughly speaking, equivalent to the condition that the connection should be closer in L^2 norm to the subspace of connections which preserve some fixed positive definite metric than any other connection in the same orbit with respect to the group of bundle automorphisms.

The proof that there is a correspondence between the zeros of this moment map and stability of connections is given in §4. The method centers on a nonlinear heat equation, in the spirit of the work of Eells and Sampson [6] on harmonic maps. In fact, one of the consequences of the main result is a classification of harmonic maps from a compact Riemannian manifold M into a negatively curved locally symmetric manifold, possibly of infinite volume. This is given in 3.5.

The ideas of Siu [14], [15] are used in the last two sections to give some further results in the special case of Kähler manifolds. As an example, at the end of §5, we show that if M is a compact Kähler manifold which has a flat $SU(n,1)$-bundle which is sufficiently nontrivial topologically, then there is a nonconstant holomorphic map from the universal cover of M into the unit ball in $\mathbf{C}^n$. In the last section, we prove a conjecture of Goldman and Millson [7] on the rigidity of actions of cocompact lattices in $SU(m,1)$ on the unit ball in $\mathbf{C}^n$. The appropriate statement for surface groups has been proven for homomorphisms with discrete, faithful image by Toledo in [16], and for homomorphisms with maximal characteristic number in [17]. The latter uses the Gromov norm of the characteristic class of the bundle to establish the result.

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2. Moment maps and connections

Let M be a compact connected Riemannian manifold, and $\pi\colon P \to M$ be a right principal $Sl(n,C)$ bundle. Suppose P has been given a Hermitian metric, i.e. a fixed reduction of the structure group to $SU(n)$. Consider the space $\mathscr{C}^\infty$ of smooth $Sl(n,C)$-connections on P. $\mathscr{C}^\infty$ is an affine space modelled on the vector space $\mathscr{E}^1(M,\mathrm{ad}(P))$ of 1-forms on M with values in the adjoint bundle associated to P. Since T^*M has a Riemannian metric, and $\mathrm{ad}(P)$ inherits a Hermitian metric from the reduction of the structure group, we have a Hermitian metric on $T^*M \otimes_{\mathbf{R}} \mathrm{ad}(P)$ which is invariant under the group $\mathscr{U}^\infty$ of smooth automorphisms of P which preserve its metric. $\mathscr{E}^1(M,\mathrm{ad}(P))$ therefore has a $\mathscr{U}^\infty$-invariant L^2 metric. We may therefore define the space of "L^2-connections" on P to be the completion of $\mathscr{C}^\infty$ with respect to the distance function induced by the norm on $\mathscr{E}^1(M,\mathrm{ad}(P))$, and we shall let it be denoted by $\mathscr{C}$. $\mathscr{C}$ is a complex Hilbert manifold, and it possesses a Hermitian metric. This metric is of course Kähler, and the Kähler form shall henceforth be denoted by ω.

The group $\mathscr{G}^k = \mathrm{Aut}(P)$ of k times continuously differentiable automorphisms of P acts on $\mathscr{C}$ by conjugation if $k \geq 1$. If, for example, D is a smooth connection on P, then g acts on D by $D \mapsto g \circ D \circ g^{-1} = Dgg^{-1}$. This extends continuously to an affine action on $\mathscr{C}$, and the unitary subgroup $\mathscr{U}^k$ preserves the Kähler structure on $\mathscr{C}$, and, a fortiori, the symplectic structure. $\mathscr{C}$ contains the affine (Lagrangian) subspace $\mathscr{A}$ of connections preserving the Hermitian metric, and we can split any connection $D \in \mathscr{C}$ into components $D = D^+ + \theta$, where $D^+ \in \mathscr{A}$ and θ is a square-integrable 1-form with values in the self-adjoint part of $\mathrm{ad}(P)$.

Proposition 2.1. *$\mathscr{C}$ is a Hamiltonian $\mathscr{U}^k$-space, with moment map given by*

$$\begin{aligned}\Phi_D(\xi) &= -i\int_M \mathrm{Tr}(D^{+,*}\theta)\xi\, d\,\mathrm{vol}\\ &= -i\langle D^{+,*}\theta, \xi\rangle,\end{aligned}$$

where $D^{+,*}$ is the adjoint of D^+ and ξ is a k times continuously differentiable skew-adjoint section of $\mathrm{ad}(P)$.

Proof. Let Ξ be the set of k times continuously differentiable skew-adjoint sections of $\mathrm{ad}(P)$. We think of Ξ as the Lie algebra of $\mathscr{U}^k$.

We need to show that Φ gives a Lie algebra homomorphism of Ξ into $C^\infty(\mathscr{C})$, where the latter is endowed with the Poisson bracket, and also that it lifts the homomorphism of Ξ into the Lie algebra of vector fields on $\mathscr{C}$ given by the group action. Let $f_\xi(A) = \Phi_A(\xi)$ with $A \in \mathscr{C}$, $\xi \in \Xi$. Then

$$\begin{aligned}(df_\xi)_A(\eta = i\eta') &= -i\int_M \operatorname{Tr}(D^{+,*}\eta - \star[i\eta', \star\theta])\xi \, d\,\mathrm{vol} \\ &= -i\langle D^{+,*}\eta - \star[i\eta', \star\theta], \xi\rangle \\ &= -i\operatorname{Im}\langle \eta, D^+\xi\rangle + \langle i\eta', [\theta, \xi]\rangle \\ &= -i\operatorname{Im}\langle \eta + i\eta', D\xi\rangle = \omega^*(D\xi, \eta + i\eta').\end{aligned}$$

Here, $\star$ is the Hodge star associated to the Riemannian structure on M, η, η' are one-forms with values in self-adjoint sections of $\mathrm{ad}(P)$, and ω^* is the symplectic inner product on $T^*\mathscr{C}$ induced by ω. We have therefore shown that Φ lifts the linear map $\Xi \to \mathrm{Ham}(\mathscr{C})$ from the Lie algebra of the unitary gauge group into Hamiltonian vector fields on the space of connections. Now consider

$$\begin{aligned}\{f_\xi, f_{\xi'}\} &= -\omega^*(df_\xi, df_{\xi'}) = -\omega(D\xi, D\xi') \\ &= -\operatorname{Im}\langle D\xi, D\xi'\rangle \\ &= -\operatorname{Im}(\langle D^+\xi, [\theta, \xi']\rangle + \langle [\theta, \xi], D^+\xi'\rangle) \\ &= -\operatorname{Im}\langle \xi, D^{+,*}[\theta, \xi'] + \star[\theta, \star D^+\xi']\rangle \\ &= -\operatorname{Im}\langle \xi, [D^{+,*}\theta, \xi']\rangle = -\operatorname{Im}\langle \xi, [\Phi_D, \xi']\rangle \\ &= -\operatorname{Im}\langle [\xi', \xi], \Phi_D\rangle = f_{[\xi, \xi']}. \quad \text{q.e.d.}\end{aligned}$$

If $D = D^+ + \theta$, then let $\widehat{D} = D^+ - \theta$. We have the following alternative method for defining Φ.

Proposition 2.2. *The differential of the function $\|\theta_D\|^2_{L^2}$ on the $\mathscr{G}$ orbit at D is $\langle 2i\Phi_D, \xi\rangle$, where the tangent space to the orbit at D has been identified with the space of sections of* $\mathrm{ad}(P)$.

Proof. Let $\Psi(g) = \|\theta_{D-Dgg^{-1}}\|^2_{L^2}$ and note that

$$\theta_{D-Dgg^{-1}} = \theta_D - \tfrac{1}{2}Dgg^{-1} - \tfrac{1}{2}g^{*,-1}\widehat{D}g^*.$$

Then

$$\begin{aligned}(d\Psi)_D(\xi) &= \langle -\tfrac{1}{2}\widehat{D}\xi^* - \tfrac{1}{2}D\xi, \theta_D\rangle + \langle \theta_D, -\tfrac{1}{2}D\xi - \tfrac{1}{2}\widehat{D}\xi^*\rangle \\ &= -2\operatorname{Re}\langle D^+\xi^+ + [\theta_D, \xi^-], \theta_D\rangle \\ &= -2\operatorname{Re}\langle \xi^+, D^{+,*}\theta_D\rangle = 2i\langle \xi^+, \Phi_D\rangle.\end{aligned}$$

Here, ξ^+ and ξ^- refer to the self-adjoint and skew-adjoint parts of ξ. q.e.d.

Thus, proving the existence of a connection which minimizes $\|\theta_D\|_{L^2}$ on a given orbit is the analogue of the problem, considered by Kempf and Ness, of finding a shortest vector in an orbit of a linear representation of G.

D will be called simple if there are no nonzero sections of $\mathrm{ad}(P)$ in its kernel.

Proposition 2.3. *If D is simple, there is at most one $\mathscr{U}$-orbit in its $\mathscr{G}$-orbit on which the moment map vanishes.*

Proof. The function Ψ defined in 2.2 can be pulled back to $i\Xi$, the vector space of self-adjoint sections of $\mathrm{ad}(P)$, by the map $\exp\colon i\Xi \to \mathscr{G}$, defined pointwise by the exponential map for $Sl(n, C)$. We define $\varphi = \Psi \circ \exp$. Since Ψ is $\mathscr{U}$-invariant, φ has a critical point if and only if Ψ does. Suppose there is a point in $i\Xi$, corresponding to a connection D, where φ has a critical point. By 2.2, $\Phi_D = 0$. We will show that this critical point is unique by showing that φ is strictly convex along any line emanating from D. Define

$$f(t) = \|\theta(\exp(t\xi))\|^2_{L^2},$$

where $\theta(\exp(t\xi))$ is the one-form occurring in the decomposition of

$$\exp(t\xi)(D) = D - D(\exp(t\xi))\exp(-t\xi).$$

We have

$$\begin{aligned}\theta(\exp(t\xi)) &= \theta - \frac{1}{2}(De^{t\xi})e^{-t\xi} - \frac{1}{2}e^{t\xi}\widehat{D}e^{t\xi} \\ &= \theta - \frac{e^{t\,\mathrm{ad}(\xi)} - 1}{2\,\mathrm{ad}(\xi)}D\xi + \frac{e^{-t\,\mathrm{ad}(\xi)} - 1}{2\,\mathrm{ad}(\xi)}\widehat{D}\xi,\end{aligned}$$

so

$$f(t) = \frac{1}{4}\|2\theta - \frac{e^{t\,\mathrm{ad}(\xi)} - 1}{\mathrm{ad}(\xi)}D\xi - \frac{e^{t\,\mathrm{ad}(\xi)} - 1}{\mathrm{ad}(\xi)}\widehat{D}\xi\|^2,$$

$$f'(t) = \frac{1}{2}\left\langle e^{t\,\mathrm{ad}(\xi)}D\xi + e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi, 2\theta - \frac{e^{t\,\mathrm{ad}(\xi} - 1}{\mathrm{ad}(\xi)}D\xi + \frac{e^{-t\,\mathrm{ad}(\xi)} - 1}{\mathrm{ad}(\xi)}\widehat{D}\xi\right\rangle,$$

$$\begin{aligned}f''(t) &= \|e^{t\,\mathrm{ad}(\xi)}D\xi + e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi\|^2 \\ &\quad - \frac{1}{2}\left\langle e^{t\,\mathrm{ad}(\xi)}\,\mathrm{ad}(\xi)D\xi - e^{-t\,\mathrm{ad}(\xi)}\,\mathrm{ad}(\xi)\widehat{D}\xi, 2\theta \right. \\ &\qquad \left. - \frac{e^{t\,\mathrm{ad}(\xi)} - 1}{\mathrm{ad}(\xi)}D\xi + \frac{e^{-t\,\mathrm{ad}(\xi)} - 1}{\mathrm{ad}(\xi)}\widehat{D}\xi\right\rangle \\ &= \frac{1}{2}\|e^{t\,\mathrm{ad}(\xi)}D\xi + e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi\|^2 + \|e^{t\,\mathrm{ad}(\xi)}D\xi - e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi\|^2 \\ &\quad + \langle e^{t\,\mathrm{ad}(\xi)}D\xi - e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi, \widehat{D}\xi - D\xi - 2[\xi,\theta]\rangle \\ &= \frac{1}{2}\|e^{t\,\mathrm{ad}(\xi)}D\xi + e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi\|^2 + \frac{1}{2}\|e^{t\,\mathrm{ad}(\xi)}D\xi - e^{-t\,\mathrm{ad}(\xi)}\widehat{D}\xi\|^2 \\ &> 0.\end{aligned}$$

Thus, any critical point is unique. q.e.d.

This proposition may be interpreted as saying that there is at most one Hermitian metric on P such that $D^{+,*}\theta = 0$, where $D^{+,*}$ and θ will depend on the choice of metric.

We will have cause to study a more general situation, so we will consider G-bundles, where G is a semisimple algebraic group over $\mathbf{R}$. Let K be a maximal compact subgroup of G, and $\pi\colon P \to M$ be a G-bundle over the compact Riemannian manifold M. We assume that we have fixed a reduction of the structure group of P to K. If D is a G-connection on P, then we have a decomposition $D = D^+ + \theta$, where D^+ is a connection preserving the K-structure and θ is a one-form on M with values in the orthogonal complement in $\mathrm{ad}(P)$ to $\mathrm{ad}(P_k)$. Here, P_K is the K-bundle defined by the reduction of structure group. We define $\Phi_D = D^{+,*}$, where $D^{+,*}$ is the adjoint to D^+ with respect to the Killing form metric. Φ may be interpreted as a moment map when G is a complex Lie group. We wish to find G-connections which are mapped by some element of $\mathrm{Aut}(P)$ to a connection for which Φ is zero. If one likes, this problem may be thought of as that of finding a K-structure which, for a fixed connection D, makes Φ_D zero. The conditions under which this may be done will be described in the next section, in the case of flat connections.

3. Harmonic metrics and flat bundles

We shall focus first on the problem of giving the proper definition of stability, and then establish the expected relationship between stable connections and zeros of the moment map.

Gauge equivalence classes of flat bundles are determined by their holonomy, and any homomorphism $\pi_1 M \to G$ determines a bundle with an equivalence class of flat connections. The space of homomorphisms of a finitely generated group such as $\pi_1 M$ into $Sl(n, C)$ can be given the structure of a complex algebraic variety. $Sl(n, C)$ acts on the variety $\mathrm{Hom}(\pi_1 M, Sl(n, C))$ by conjugation. The machinery of geometric invariant theory applies, and one finds that the stable representations are those whose image is not contained in any nontrivial parabolic subgroup of $Sl(n, C)$. We take the analogous condition for connections as our definition of stability. Suppose E is the vector bundle associated to P by the standard representation of $Sl(n, C)$.

Definition 3.1. *Let D be a flat connection on P. It is stable if E has no nontrivial D-invariant subbundles. It will be called reductive if any D-invariant subbundle has a D-invariant complement.*

Any reductive connection is a direct sum of stable ones. In §2, it was shown that there is at most one $\mathscr{U}$-orbit in the $\mathscr{G}$-orbit of D on which the moment map is zero, provided D is simple. The following is a nonexistence result for zeros of the moment map. In the sequel, it will be shown that zeros do exist in the case of a reductive connection.

Proposition 3.2. *Suppose D is not reductive. Then Φ is nonzero everywhere on the $\mathcal{G}$-orbit $\mathcal{O}_D$ of D.*

Proof. Since D is not reductive, there is a D-invariant subbundle E_1 of E, and a corresponding D-invariant quotient $E_2 = E/E_1$. E_2 is isomorphic as a vector bundle to the orthogonal complement of E_1 in E, and we shall henceforth identify the two. Due to the nonreductivity, we may assume that the splitting $E = E_1 \oplus E_2$ is not D-invariant. Relative to this decomposition, D takes the following form:

$$\begin{pmatrix} D_1 & \eta \\ 0 & D_2 \end{pmatrix},$$

where D_1 is the connection on E_1 given by $p_1 \circ D$, D_2 is the connection $p_2 \circ D$ on E_2, p_1, p_2 are the orthogonal projections of E on E_1 and E_2, and η is a nonzero one-form on M with values in $\mathrm{Hom}(E_2, E_1)$. Let n_i be the rank of E_i, and define $\xi = n_2|_{E_1} \oplus (-n_1)|_{E_2}$. Then ξ is a self-adjoint section of $\mathrm{ad}(P)$, and

$$D - D(e^{t\xi})e^{-t\xi} = \begin{pmatrix} D_1 & e^{t(n_1+n_2)}\eta \\ 0 & D_2 \end{pmatrix}.$$

Furthermore,

$$\|\theta(e^{t\xi}(D))\|_2^2 = \left\| \begin{pmatrix} \theta_1 & \frac{1}{2}e^{t(n_1+n_2)}\eta \\ \frac{1}{2}e^{t(n_1+n_2)}\eta^* & \theta_2 \end{pmatrix} \right\|_2^2.$$

This function has a finite limit as t approaches $-\infty$, and it is strictly convex, so it has no critical points. In particular, D is not a zero of Φ. The same argument applies to any equivalent connection. q.e.d.

The following theorem will be proved in the next section.

Theorem 3.3. *If D is a stable flat connection, there is a unique $\mathcal{U}$-orbit in $\mathcal{O}_D$ on which Φ vanishes. Equivalently, there is a unique metric on P for which the corresponding value of Φ is zero.*

We shall call metrics which satisfy the condition of the theorem harmonic. The motivation for this name may be explained as follows. Since D is flat, the bundle of metrics on the pullback of P to the universal cover of M is canonically equivalent to the product $\tilde{M} \times Sl(n,C)/SU(n)$, up to the action of elements of $Sl(n,C)$. A metric on P is a $\pi_1 M$-equivariant map H from $\tilde{M}$ to $Sl(n,C)/SU(n)$. The condition $\Phi = 0$ is equivalent to H being harmonic as a map of Riemannian manifolds. This may be seen by observing that the one-form θ is identical with the differential of H and D^+ is the pullback of the canonical Riemannian connection on $Sl(n,C)SU(n)$.

Everything we have done so far, and all that will be done in the following section, remains valid if we restrict to a real semisimple Lie subgroup G of $Sl(n,C)$. In this paragraph and the next, the notation will be that of the end of the previous section. We shall call a flat connection D on a principal G-bundle P stable if its holonomy at any point is not contained in a nontrivial parabolic subgroup of G. It is reductive if the Zariski closure of its holonomy at any point is a reductive subgroup of G. From 3.2 and the result of Birkes [2] that a destabilizing one-parameter subgroup may be taken to be defined over $\mathbf{R}$, we get the analogue of 3.2 for flat G-bundles. We also get the analogue of 3.3 from the invariance of the argument to be given in the next section under restriction to G. Thus, we have

Theorem 3.4. *If D is a stable flat connection on the G-bundle P, there is a unique K-structure on P for which the corresponding value of Φ is zero.*

Again, we shall use the phrase 'harmonic metric' to describe such a K-structure.

As a special case of 3.4, we obtain an extension of a result of Eells and Sampson on the existence of harmonic maps. In [6], they proved that any homotopy class of maps $M \to N$, where N is compact and has nonpositive sectional curvature, has a harmonic representative. We can give an extension of this to the case where N is a noncompact locally symmetric manifold. The following is a corollary of 3.4.

Corollary 3.5. *Suppose G is a real semisimple algebraic group, K is a maximal compact subgroup, and N is a Riemannian manifold (possibly of infinite volume), which is covered by G/K. If M is a compact Riemannian manifold and ν is a homotopy class of maps from M to N, then ν has a harmonic representative if and only if the Zariski closure of $\nu_*\pi_1 M$ in G is reductive.*

This result makes it plausible that there should be a reasonable way of classifying harmonic maps with noncompact negatively curved targets. For example, if N is a hyperbolic manifold, then one can express the reductivity condition on $\nu_*\pi_1 M$ as a condition on the fixed point set for the action of $\pi_1 M$ on the sphere at infinity associated with hyperbolic space. One has a similar sphere at infinity associated with any negatively curved manifold, and it might be possible, in some cases, to give a similar condition which would ensure the existence of a harmonic map in a given homotopy class. Unfortunately, we are unable to pursue this line of enquiry here.

4. The existence thereom

This section is devoted to the proof of 3.3. The method is in the same genre as the arguments in [5] and [6]. We attempt to reform a given flat connection by means of a nonlinear heat equation, which can be given in the following two equivalent forms:

$$\frac{\partial A}{\partial t} = -D_A \Phi_A, \tag{1}$$

$$\frac{\partial g}{\partial t} = -\Phi_g g. \tag{2}$$

Here, A is a time-dependent family of connections in the orbit of D, g is a time-dependent family of elements of $\mathscr{G}$, Φ_A is $D_A^{+,*}\theta_A$, and Φ_g is $\Phi_{g(D)}$. The proof falls into four parts:

1. The existence of a solution to the equations above for a short period of time;

2. existence of a solution for $0 \le t < \infty$;

3. convergence of the solution to (1) at $t = \infty$; and

4. investigation of the dichotomy between the case where the limiting connection lies in the initial orbit and that where it does not.

The first step is by now fairly standard. One need only arrange things so that an inverse function can be applied. We refer to [10] for the details of the argument.

Now assume we have a solution of the evolution equation defined on a maximal time interval $[0, T)$. The interval of definition is necessarily open on the right end, since we could otherwise use the short time existence to extend the solution beyond T. We shall, for the time being, use the first version of the evolution equation, so A will be a time-dependent connection with the original choice of connection as its initial value. D will be the associated covariant derivative, and $D = D^+ + \theta$ the usual decomposition. Δ will be the operator d^*d on functions and $\Box^+$ will be the operator $D^+D^{+,*} + D^{+,*}D^+$. We shall need the usual array of Sobolev (denoted by L_k^p) and C^0 spaces of section of vector bundles. If f is some section of a vector bundle over $M \times [0, T)$, then $\|f\|_p$ will be its L^p-norm on M, considered as a function of time, $\|f\|_{p,k}$ will be its L_k^p-norm on M, and $\|f\|_{C^0}$ its C^0 norm. $(\ ,\ )$ will be the L^2 inner product on M, $\langle\ ,\ \rangle$ the pointwise orthogonal inner product on $M \times [0, T)$, and $|\ |$ the pointwise norm. $\mathscr{C}^{p,k}$ will be the completion of $\mathscr{C}$ with respect to an L_k^p-norm and, if $pk > m = \dim M$, then $\mathscr{G}_k^p$ will be the completion of the gauge group in the same norm. Positive constants independent of time shall frequently enter into our calculations, and they shall be denoted by C, C', etc. Two occurrences of such symbols should not be expected to refer to the same constant if they are separated by any bits of ordinary text.

Because D is flat, we have

$$\Omega^+ + \tfrac{1}{2}[\theta, \theta] = 0, \qquad D^+\theta = 0.$$

Here, Ω^+ is the curvature of D^+. Now we have

$$\frac{d}{dt}\|\theta\|_2^2 = -2(D^+\Phi, \theta) = -2\|\Phi\|_2^2 \leq 0,$$

$$\begin{aligned}\frac{\partial}{\partial t}|\Phi|^2 = \frac{\partial}{\partial t}|D^{+,*}\theta|^2 &= -2\langle D^{+,*}D^+\Phi - \star[[\theta,\Phi],\star\theta],\Phi\rangle \\ &= -2\langle \Box^+\Phi,\Phi\rangle - 2|[\theta,\Phi]|^2 \\ &= -\Delta|\Phi|^2 - 2|D^+\Phi|^2 - 2|[\theta,\Phi]|^2.\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}|\Phi|^2 + \Delta|\Phi|^2 \leq 0.$$

In particular, the maximum principle implies that $\|\Phi\|_{C^0}$ is bounded uniformly on $[0,T)$. Also, from the first calculation, $\|\theta\|_2$ is bounded uniformly for all time.

Next, we need a Weitzenböck formula for $\Box^+$ on one-forms with values in the self-adjoint part of $\mathrm{ad}(P)$. Let ∇ be the connection on $T^*M \otimes \mathrm{ad}(P)$ constructed from D^+ and the canonical Riemannian connection on T^*M. Choose a normal coordinate system at $p \in M$ and let e_i be an orthonormal frame in T_pM, extended to vector fields in a neighborhood of p by parallel transport along geodesics. Let ϕ be a one-form with values in $i\Xi(P)$. Calculating at p, we find

$$\begin{aligned}\Box^+\phi(e_i) &= \nabla_i(D^{+,*}\phi) - \sum_{j=1}^{n}\nabla_j(D^+\phi)(e_j,e_i) \\ &= -\nabla_i\sum_{j=1}^{n}(\nabla_j\phi)(e_j) - \sum_{j=1}^{n}\nabla_j(\nabla_j\phi_i - \nabla_i\phi_j) \\ &= \nabla^*\nabla\phi(e_i) - \sum_{j=1}^{n}\Omega_{ij}^{\nabla}\phi(e_j) \\ &= \nabla^*\nabla\phi(e_i) - \sum_{j=1}^{n}\phi(\Omega_{ji}^{TM}e_j) + \sum_{j=1}^{n}[\Omega_{ji}^+,\phi_j] \\ &= \nabla^*\nabla\phi(e_i) + \phi\circ \mathrm{Ricci}_M(e_i) - \frac{1}{2}\sum_{j=1}^{n}[[\theta_j,\theta_i],\phi_j] \\ &= \nabla^*\nabla\phi(e_i) + (\phi\circ\mathrm{Ricci}_M)(e_i) + \frac{1}{4}\star[\phi,\star[\theta,\theta]](e_i).\end{aligned}$$

Ricci_M is the Ricci curvature of M, and it will be denoted by Ric from now on. Just now, we want the following special case of this formula:

$$\begin{aligned}\langle\Box^+\theta,\theta\rangle &= \langle\nabla^*\nabla\theta,\theta\rangle + \langle\theta\circ\mathrm{Ric},\theta\rangle + \tfrac{1}{4}|[\theta,\theta]|^2 \\ &= \tfrac{1}{2}\Delta|\theta|^2 + |\nabla\theta|^2 + \langle\theta\circ\mathrm{Ric},\theta\rangle + \tfrac{1}{4}|[\theta,\theta]|^2.\end{aligned}$$

Thus,

$$\|\Phi\|_2^2 = (\Box^+\theta, \theta) = \|\nabla\theta\|_2^2 + (\theta \circ \mathrm{Ric}, \theta) + \tfrac{1}{4}\|[\theta, \theta]\|_2^2,$$

which implies

$$\|\nabla\theta\|_2^2 + \|[\theta, \theta]\|_2^2 \leq C(\|\Phi\|_2^2 + \|\theta\|_2^2) \leq C'$$

for all $t \in [0, T)$.

The following is an application of the Sobolev inequalities. For $p \geq 1$,

$$\begin{aligned} \|\theta\|_{2mp/(m-2)} &\leq C \int_M |d(1+|\theta|^2)^{p/2}|^2 \, d\mathrm{vol} \\ &= C \int_M |p\langle\nabla\theta, \theta\rangle(1+|\theta|^2)^{p/2-1}|^2 \, d\mathrm{vol} \\ &\leq C' \int_M |\nabla\theta|^2 (1+|\theta|^2)^{p-1} \, d\mathrm{vol}. \end{aligned} \tag{3}$$

We calculate as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}(1+|\theta|^2)^p \\ &\quad = -2p(1+|\theta|^2)^{p-1}\langle\Box^+\theta, \theta\rangle \\ &\quad = -2p(1+|\theta|^2)^{p-1}\left[\frac{1}{2}\Delta|\theta|^2 + |\nabla\theta|^2 + \langle\theta \circ \mathrm{Ric}, \theta\rangle + \frac{1}{4}|[\theta, \theta]|^2\right], \end{aligned} \tag{4}$$

$$\begin{aligned} \Delta(1+|\theta|^2)^p &= d^*[p(1+|\theta|^2)^{p-1} \, d|\theta|^2] \\ &= -p(p-1)(1+|\theta|^2)^{p-2}|d|\theta|^2|^2 \\ &\quad + \Delta p(1+|\theta|^2)^{p-1}\Delta|\theta|^2. \end{aligned} \tag{5}$$

Adding (4) and (5) gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right)(1+|\theta|^2)^p = &- p(p-1)(1+|\theta|^2)^{p-2}|d|\theta|^2|^2 \\ &- 2p(1+|\theta|^2)^{p-1}[|\nabla\theta|^2 + \langle\theta \circ \mathrm{Ric}, \theta\rangle + \tfrac{1}{4}|[\theta, \theta]|^2] \\ \leq &- 2p(1+|\theta|^2)^{p-1}|\nabla\theta|^2 + C(1+|\theta|^2)^{p-1}|\theta|^2. \end{aligned}$$

Integrating over M and rearranging the terms give

$$\int_M (1+|\theta|^2)^{p-1}|\nabla\theta|^2 \, d\mathrm{vol} \leq C\left|\frac{d}{dt}\|1+|\theta|^2\|_p^p\right| + C'\|1+|\theta|^2\|_p^p. \tag{6}$$

The first term on the right can be estimated as follows:

$$\begin{aligned}
\left|\frac{d}{dt}\|1+|\theta|^2\|_p^p\right| &= 2p\left|\int_M (D^+\Phi, (1+|\theta|^2)^{p-1}\theta)\, d\,\mathrm{vol}\right| \\
&= 2p\left|\int_M (\Phi, D^{+,*}[(1+|\theta|^2)^{p-1}\theta])\, d\,\mathrm{vol}\right| \\
&= 2p\left|\int_M ((\Phi, (1+|\theta|^2))^{p-1}\Phi)\right. \\
&\qquad \left. - \star(2p-2)[(1+|\theta|^2)^{p-2}(\nabla\theta,\theta)\wedge\star\theta]\right| \, d\,\mathrm{vol} \\
&\le C\int_M (1+|\theta|^2)^{p-1}|\nabla\theta| d\,\mathrm{vol} + C\int_M (1+|\theta|^2)^{p-1}\, d\,\mathrm{vol}.
\end{aligned} \tag{7}$$

The fact that $\|\Phi\|_{C^0}$ is uniformly bounded has been used to get the last line. (6) and (7) imply

$$\int_M (1+|\theta^2)^{p-1}|\nabla\theta|^2\, d\,\mathrm{vol} \le C\left[\int_M (1+|\theta|^2)^{p-1}|\nabla\theta|\, d\,\mathrm{vol} + \|1+|\theta|^2\|_p^p\right]. \tag{8}$$

Let $k \ge 2$ be an integer. From (3), (8) and Hölder's inequality, we get

$$\|\theta\|_{2mp/(m-2)} \le C[1 + \|\theta\|_{4p-2k}^{2p-k}\|\, |\theta|^{k-2}|\nabla\theta|\, \|_2 + \|\theta\|_{2p}^{2p}]. \tag{9}$$

(3) and (9) may be used as moves in an iterative process which eventually gives a bound on $\|\theta\|_q$ for any value of q. The process begins by taking (9) with $k = 2$:

$$\|\theta\|_{2mp/(m-2)} \le C[1 + \|\theta\|_{4p-4}^{2p-2}\|\nabla\theta\|_2 + \|\theta\|_{2p}^{2p}].$$

In case $p = 1$,

$$\|\theta\|_{2m/(m-2)} \le C[1 + \|\nabla\theta\|_2 + \|\theta\|_2^2] \le C'.$$

Next take $p = \min(2, 2m/(m-2))$. Then (9) again gives a bound on $\|\theta\|_{\frac{2mp}{m-2}}$. This may be repeated until $p = 2$, which gives

$$\|\theta\|_{4m/(m-2)} \le C[1 + \|\theta\|_4^2\|\nabla\theta\|_2 + \|\theta\|_4^4] \le C'.$$

From (8), we have also gained the following bound:

$$\int_M (1+|\theta|^2)|\nabla\theta|^2\, d\,\mathrm{vol} \le C.$$

Now (9) becomes useful for $k = 3$ and $2 < p \le 3$. We may use it repeatedly until we have obtained uniform bounds on

$$\|\theta\|_{6m/(m-2)}, \quad \int_M (1+|\theta|^2)^2|\nabla\theta|^2\, d\,\mathrm{vol},$$

just as above. Now the version of (9) with $k = 4$ and $3 < p \leq 4$ becomes relevant, and the process continues as long as necessary to obtain a bound on $\|\theta\|_{2p}$ for some $p > m$.

First, we use this to show that the solution to the heat equation extends till the end of time. As t approaches T, observe that $\|\Omega^+\|_p = \frac{1}{2}\|[\theta,\theta]\|_p$ is uniformly bounded. We now quote a theorem of Uhlenbeck, given in [18].

Theorem 4.1. *Let D_i be a sequence of unitary connections on a bundle over a compact Riemannian manifold M such that $\|\Omega_i\|_p < B$, $1 \leq i < \infty$, $2p > \dim M$. Then there exists a sequence u_i of L^p_2 unitary automorphisms of the bundle such that the sequence $D_i - D_i u_i u_i^{-1}$ lies in a bounded subset of $\mathscr{C}^{p,1}$.*

In particular, there is some family u_t such that $D^{+,\prime} = D^+ - D^+uu^{-1}$ remains in a bounded subet of $\mathscr{C}^{p,1}$. Define $\eta_t = D_t^{+,\prime}$, and $\theta_t' = u\theta u^{-1}$. We know that

$$D_t^{+,\prime}\theta_t' = D_0^{+,\prime}\theta_t + [\eta_t, \theta_t'] = 0.$$

Furthermore, η_t is uniformly bounded in L^p_1-norm, so the Sobolev embedding theorem assures us of a uniform bound on the L^∞-norm of η_t. Since $\|\theta_t'\|_p$ is uniformly bounded, we get a bound on the L^p-norm of $[\eta_t, \theta_t']$, so $\|D_0^{+,\prime}\theta_p'\|$ is uniformly bounded. Similarly, the fact that

$$\Phi_t' = D_t^{+,\prime,*}\theta_t' = D_0^{+,\prime,*}\theta_t' - \star[\eta_t, \star\theta_t'],$$

and $\|\Phi'\|_p$ is uniformly bounded implies that $\|D_0^{+,\prime,*}\theta'\|_p$ is uniformly bounded. Hence, θ' is uniformly bounded in the L^p_1-norm. This implies that $D'g'g'^{,-1}$ is bounded in the L^p_1-norm, where $D' = D^{+,\prime} + \theta'$ and $g' = ug$.

Observe that

$$\frac{\partial}{\partial t}|g|^2 = -2\langle\Phi_g g, g\rangle \leq C|g|^2,$$

so $|g|^2 \leq Ae^{Ct}$, and $\|g\|_{C^0}$ is uniformly bounded on any finite time interval. The same holds true for $\|g'\|_{C^0}$, since $|g'| = |g|$. $\|D'g'g'^{,-1}\|_p < C$ implies $\|D'g'\|_p < C'$, so $\|g'\|_{p,1} < C''$. Similarly, $\|D'g'g'^{,-1}\|_{p,1} < C'''$ implies $\|g\|_{p,2} < C''''$ on $[0,T)$. Thus, the solution of (2) extends to $[0,T]$, and, by the short time existence, to some $[0, T+\varepsilon)$. Since T was assumed maximal, the solution exists for all time.

Now consider the family of connections D_t' as t approaches ∞. We know that they lie in a weakly compact subset of $\mathscr{C}^{p,1}$, so some subsequence converges weakly to a connection D_∞. We claim that if D_0 was stable, then D_∞ must lie in the same $\mathscr{G}^p_2$ orbit. To show this, we borrow an argument from Donaldson [4]. Note first that, by standard arguments, the L^p-norm of the curvature is a weakly lower semicontinuous function on $\mathscr{C}^{p,1}$. Hence, D_∞ is flat. We argue that $\mathrm{Hom}(D_0, D_\infty)$ is nonzero, i.e. there is a nontrivial section σ of $\mathrm{Hom}(E,E)$ for which $D_{0,\infty}\sigma = 0$, where $D_{0,\infty}$ is the connection on $\mathrm{Hom}(E,E)$ induced by D_0 on the first factor and D_∞ on the second. If $\mathrm{Hom}(D_0, D_\infty) = 0$, then

$$\|D_{0,\infty}\sigma\|_m \geq C\|\sigma\|_{2m}, \qquad \sigma \in \mathrm{Hom}(E,E).$$

By the embedding theorem for Sobolev spaces, we have

$$\|D_{0,\infty}\sigma\|_m \geq C'\|\sigma\|_{2m}.$$

Furthermore, the embedding $L_1^m \to L^{2m}$ is compact, so D_t' converges to D_∞ in the L^{2m}-norm. Let $\phi_t = D_\infty - D_t'$. Then

$$\begin{aligned}\|D_{0,\infty}\sigma\|_m - \|D_{0,t}\sigma\|_m &\leq \|D_{0,\infty}\sigma - D_{0,t}\sigma\|_m \\ &\leq C''\|\phi_t\sigma\|_m \leq C''\|\phi_t\|_{2m}\|\sigma\|_{2m}.\end{aligned}$$

For each t and σ, one has

$$\begin{aligned}\|D_{0,t}\sigma\|_m &\geq \|D_{0,\infty}\sigma\|_m - C''\|\phi_t\|_{2m}\|\sigma\|_{2m} \\ &\geq (C' - C''\|\phi_t\|_{2m})\|\sigma\|_{2m}.\end{aligned}$$

We may make $\|\phi_t\|_{2m}$ as small as we like by choosing t close enough to ∞, so $C' - C''\|\phi_t\|_{2m}$ is eventually positive. But this implies that $\mathrm{Hom}(D_0, D_t') = 0$ for some t, contradicting the fact that the D' all lie in the same orbit for $t \in [0,\infty)$. Thus, there must be a nontrivial element σ of $\mathrm{Hom}(D_0, D_\infty)$. Since D_0 and D_∞ are both flat and σ is a covariant constant, its rank must be constant on M. Of necessity, the kernel and cokernel of σ must be D_0 and D_∞ invariant, respectively. However, D_0 preserves no nontrivial subbundles, so σ has zero as its kernel, and is therefore an isomorphism of D_0 and D_∞. Thus, D_∞ lies in the same orbit as D_0. The corresponding value of Φ is zero. Smoothness of harmonic metrics associated with smooth connections follows by standard regularity arguments. The proof of 3.3 is complete.

5. Holomorphic metrics and Siu's argument

In [14] and [15], Siu gave criteria for harmonic maps between Kähler manifolds to be holomorphic. Results of this sort are of interest in the context of flat G-bundles over Kähler manifolds when G/K is a Hermitian symmetric space. It makes sense in the situation to speak of holomorphic metrics, and these will be special cases of harmonic metrics. We shall exploit Siu's methods to give criteria for flat $SU(n,1)$-bundles over compact Kähler manifolds to admit holomorphic metrics. All the essential moves in the argument are found in [14] and [15], although a few minor modifications have been found necessary in order to apply them in the current situation. The principal innovation is in the applications, which exploit 3.4.

Suppose that M is Kähler with Kähler form ω, and P is a principal G-bundle over M with flat connection D. We assume that G/K is a Hermitian symmetric space. We preserve the notation of the previous sections. The usual decomposition of $T^*M \otimes C$ into (1,0) and (0,1) components is available. Assume that P has a harmonic metric, with $D = D^+ + \theta$ the corresponding decomposition. Let $D^+ = \partial^+ + \overline{\partial}^+$, with ∂^+ the composition of D^+ with projection onto (1,0)-forms and $\overline{\partial}^+$ the composition of D^+ with projection onto (0,1)-forms. We also have $\mathrm{ad}(P) = \mathbf{V} \oplus \mathbf{W}$, corresponding to the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ of the Lie algebra of G. Since G/K is Hermitian symmetric, there is the further decomposition $\mathbf{p} \otimes \mathbf{C} = \mathbf{p}^+ \oplus \mathbf{p}^-$ into the eigenspaces corresponding to i and $-i$ of the complex structure J. The corresponding decomposition of $\mathbf{W}$ is $\mathbf{W}^+$ and $\mathbf{W}^-$. We have the following fourfold decomposition:

$$\theta = \theta_+^{1,0} + \theta_-^{1,0} + \theta_+^{0,1} + \theta_-^{0,1}.$$

Note that $\theta_-^{0,1}$ is the complex conjugate of $\theta_+^{1,0}$, and $\theta_+^{0,1}$ that of $\theta_-^{1,0}$. Observe further that the metric is holomorphic if and only if $\theta_+^{0,1}$ is zero, and it is antiholomorphic if $\theta_-^{0,1}$ is zero. From the condition $D^+\theta = 0$, we get

$$\partial^+\theta_+^{1,0} = 0, \quad \overline{\partial}^+\theta_+^{0,1} = 0, \quad \partial^+\theta_-^{1,0} = 0, \quad \overline{\partial}^+\theta_-^{0,1} = 0,$$
$$\overline{\partial}^+\theta_+^{1,0} + \partial^+\theta_+^{0,1} = 0, \qquad \overline{\partial}^+\theta_-^{1,0} + \partial^+\theta_-^{0,1} = 0.$$

From the fact that the curvature of D^+ is $-\frac{1}{2}[\theta,\theta]$, and $\mathbf{p}^+$ and $\mathbf{p}^-$ are abelian Lie algebras, we get

$$\partial^+ \circ \partial^+ = -[\theta_+^{1,0}, \theta_-^{1,0}], \qquad \overline{\partial}^+ \circ \overline{\partial}^+ = -[\theta_+^{0,1}, \theta_-^{0,1}].$$

To formulate the condition that the metric is harmonic in a convenient form, we shall need to calculate in a simply connected neighborhood U of $p \in M$. We have the Kähler identities for $\partial^{+,*}$ and $\overline{\partial}^{+,*}$:

$$\partial^{+,*} = -i[L^*, \overline{\partial}^+], \qquad \overline{\partial}^{+,*} = i[L^*, \partial^+].$$

Note that, in a neighborhood such as we have chosen, we can think of the harmonic metric as a map $f\colon U \to G/K$, and $\theta = -\frac{1}{2}\, df f^{-1}$, where we have identified G/K with self-adjoint elements of G for some embedding in $Sl(n,C)$, and we use a metric which is parallel for d as the background metric. Then

$$\begin{aligned}\partial^{+,*}\theta^{1,0} &= \tfrac{1}{2} i L^* \overline{\partial}^+[\partial f f^{-1}] \\ &= \tfrac{1}{2} i L^* (\overline{\partial}[\partial f f^{-1}] + \tfrac{1}{2}[\overline{\partial} f f^{-1}, \partial f f^{-1}]).\end{aligned}$$

Similarly,

$$\begin{aligned}
2D^{+,*}\theta &= iL^*(\partial^+ - \overline{\partial}^+)[\partial f f^{-1} + \overline{\partial} f f^{-1})\\
&= iL^*(\partial^+[\overline{\partial} f f^{-1}] - \overline{\partial}^+[\partial f f^{-1}])\\
&= iL^*(\partial[\overline{\partial} f f^{-1}] - \overline{\partial}[\partial f f^{-1}] + \tfrac{1}{2}[\partial f f^{-1}, \overline{\partial} f f^{-1}]\\
&\qquad\qquad - \tfrac{1}{2}[\overline{\partial} f f^{-1}, \partial f f^{-1}])\\
&= iL^*(\overline{\partial} f f^{-1}\partial f f^{-1} + \partial\overline{\partial} f f^{-1} - \overline{\partial}[\partial f f^{-1}])\\
&= iL^*(\overline{\partial} f f^{-1}\partial f f^{-1} - \overline{\partial}\partial f f^{-1} - \overline{\partial}[\partial f f^{-1}])\\
&= iL^*(\partial f f^{-1}\overline{\partial} f f^{-1} + \overline{\partial} f f^{-1}\partial f f^{-1} - 2\overline{\partial}[\partial f f^{-1}])\\
&= iL^*(-2\overline{\partial}[\partial f f^{-1}] + [\overline{\partial} f f^{-1}, \partial f f^{-1}]) = 4\partial^{+,*}\theta^{1,0}.
\end{aligned}$$

Thus, f is a harmonic metric if and only if $\partial^{+,*}\theta^{1,0} = 0$, i.e., if and only if $\overline{\partial}^+\theta^{1,0}$ is a primitive $(1,1)$-form.

Let $\langle\ ,\ \rangle_{\mathbf{W}}$ be the complexification of the pointwise K-invariant orthogonal inner product on $\mathbf{W}$. Let $(\ ,\ )$ be the pointwise metric on $\mathbf{W}\otimes\mathbf{C}$ defined by $(w, w') = \langle w, \bar{w}'\rangle_{\mathbf{W}}$, or on any bundle of the form $\bigwedge^p T^*M \otimes \mathbf{W}\otimes\mathbf{C}$. The next result is our version of Siu's Bochner type formula for harmonic maps.

Theorem 5.1. *If $D = D^+ + \theta$ is the decomposition associated with a harmonic metric on P, then $[\theta_+^{1,0}, \theta_-^{1,0}] = 0$.*

Proof. First we calculate

$$\begin{aligned}
\partial\overline{\partial}\langle\theta_+^{0,1}, \theta_-^{1,0}\rangle_{\mathbf{W}} &= -\langle\partial^+\theta_+^{0,1}, \overline{\partial}^+\theta_-^{1,0}\rangle_{\mathbf{W}} + \langle\theta_+^{0,1}, \partial^+\overline{\partial}^+\theta_-^{1,0}\rangle_{\mathbf{W}}\\
&= -\langle\partial^+\theta_+^{0,1}, \overline{\partial}^+\theta_-^{1,0}\rangle_{\mathbf{W}} - \langle\theta_+^{0,1}, \partial^+\partial^+\theta_-^{0,1}\rangle_{\mathbf{W}}\\
&= -\langle\partial^+\theta_+^{0,1}, \overline{\partial}^+\theta_-^{1,0},\rangle_{\mathbf{W}} + \langle\theta_+^{0,1}, [[\theta_+^{1,0}, \theta_-^{1,0}], \theta_-^{0,1}]\rangle_{\mathbf{W}}\\
&= -\langle\partial^+\theta_+^{0,1}, \overline{\partial}^+\theta_-^{1,0}\rangle_{\mathbf{W}} - \langle[\theta_+^{0,1}, \theta_-^{0,1}], [\theta_+^{1,0}, \theta_-^{1,0}]\rangle_{\mathbf{W}}.
\end{aligned}$$

Since $\overline{\partial}^+\theta_-^{1,0}$ is primitive, we have

$$-\frac{1}{(n-2)!}\langle\partial^+\theta_+^{0,1}, \overline{\partial}^+\theta^{1,0}\rangle_{\mathbf{W}} \wedge \omega^{m-2} = (\partial^+\theta_+^{0,1}, \partial^+\theta_+^{0,1})\omega^m,$$

as a consequence of Theorem 2 on p. 23 of [20]. Thus, the left side is a nonnegative $2n$-form. Wedging the result of the initial calculation with ω^{m-2} gives an exact form as a sum of two nonnegative $2n$-forms. Hence, each of the three must be zero. In particular, this implies $[\theta_+^{1,0}, \theta_-^{1,0}] = 0$. q.e.d.

From this point on, we will specialize to the case $G = SU(n,1)$. For future reference, the reader should keep in mind that, with minor modifications, the following arguments apply to $PSU(n,1)$, which may be realized as a linear group by means of the adjoint representation.

Proposition 5.2. *Suppose P is a flat $SU(n,1)$-bundle with harmonic metric. If θ has real rank at least four as a map $\theta\colon T_pM \to \mathbf{W}_p$ at some $p \in M$, then $\theta_+^{1,0}$ or $\theta_-^{0,1}$ vanishes at p.*

Proof. Let $V = T_pM \otimes \mathbf{C}$ and identify $\mathbf{W}_p \otimes \mathbf{C}$ with $\mathbf{p_C} = \mathbf{p} \otimes \mathbf{C}$. $\theta^{1,0}\colon V \to \mathbf{p_C}$ has complex rank at least two, so there exist two dimensional subspaces of V on which the restriction of $\theta^{1,0}$ is injective. Suppose that v_1 and v_2 are a basis for one such subspace. Let v_1^* and v_2^* be the dual basis. Then the restriction of $\theta^{1,0}$ to this subspace takes the form

$$\theta^{1,0} = v_1^* \otimes p_1 + v_2^* \otimes p_2,$$

with p_1 and p_2 linearly independent in $\mathbf{p_C}$. Let $p_1 = p_1^+ + p_1^-$ and $p_2 = p_2^+ + p_2^-$ be the decompositions according to the direct sum $\mathbf{p_C} = \mathbf{p}^+ \oplus \mathbf{p}^-$. On the subspace in question, we have

$$[\theta^{1,0}, \theta^{1,0}] = v_1^* \wedge v_2^* \otimes [p_1, p_2] = v_1^* \wedge v_2^* \otimes ([p_1^+, p_2^-] + [p_1^-, p_2^+]).$$

One has a natural isomorphism of $\mathbf{p}^+$ with the matrices in $sl(n+1, C)$ whose entries are zero unless they are simultaneously in the last column and first n rows. Similarly, $\mathbf{p}^-$ can be identified with the Lie algebra of transposes of such matrices. Assume $\theta_+^{1,0} \neq 0$. We may choose coordinates in $\mathbf{p}^+$ so that

$$p_1^+ = \begin{pmatrix} \cdots & 0 & 1 \\ \cdots & 0 & 0 \\ & & \vdots \end{pmatrix}, \qquad p_1^- = \begin{pmatrix} \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ a_1 & \cdots & a_n & 0 \end{pmatrix},$$

$$p_2^+ = \begin{pmatrix} \cdots & 0 & b_1 \\ & \vdots & \vdots \\ \cdots & 0 & b_n \\ \cdots & 0 & 0 \end{pmatrix}, \qquad p_2^- = \begin{pmatrix} \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ c_1 & \cdots & c_n & 0 \end{pmatrix}.$$

Then

$$[p_1^+, p_2^-] = \begin{pmatrix} c_1 & \cdots & c_n & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & -c_1 \end{pmatrix},$$

$$[p_1^-, p_2^+] = \begin{pmatrix} & & & 0 \\ & -b_i a_j & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \sum_{k=1}^n a_k b_k \end{pmatrix}.$$

Suppose some $c_j \neq 0$. Then $c_j = b_1 a_j \neq 0$, so $a_j \neq 0$. If $a_j \neq 0$, then $b_i = 0$ for $2 \leq i \leq n$, so $p_2^+ = b_1 p_1^+$.

$$[p_1^+, p_2^-] + [p_1^-, p_2^+] = [p_1^+, p_2^- - b_1 p_1^-] = 0.$$

This implies $p_2^- - b_1 p_1^- = 0$, so $p_2 = b_1 p_1$ and $\theta^{1,0}$ is not injective on the subspace generated by v_1 and v_2. Since we obtain this contradiction if any of $a_1, \cdots, a_n, c_1, \cdots, c_n$ is nonzero, it must be that $\theta_-^{1,0}$ is zero on the subspace generated by v_1 and v_2.

The set of two dimensional subspaces of V on which $\theta^{1,0}$ is injective is an open dense subset of the Grassmannian of 2-planes in V. The set of 2-planes for which the restriction of $\theta_+^{1,0}$ vanishes is a component of this subset, and similarly for those on which $\theta_-^{1,0}$ vanishes. Thus, the restriction of one of the two to 2-planes in V vanishes on a nonempty open subset of the Grassmannian, so one of them must be identically zero on V. q.e.d.

The main result follows from Aronszajn's unique continuation theorem for elliptic systems, exactly as in Proposition 4 of [14].

Theorem 5.3. *If P is a flat $SU(n,1)$-bundle over M with harmonic metric and θ has real rank four at some $p \in M$, then the metric is either holomorphic or antiholomorphic.*

A maximal compact subgroup of $SU(n,1)$ is isomorphic to $U(n)$, so we may associate Chern classes $c_i(P)$ with any $SU(n,1)$-bundle P. The Chern forms constructed from the curvature of D^+ are locally pullbacks of the canonical Chern forms on complex hyperbolic space, and the semisimplification of a flat bundle is topologically equivalent to it, so we have the following corollary of the theorem.

Corollary 5.4. *If P is a flat $SU(n,1)$-bundle over M with $c_i(P)$ topologically nonzero for some $i \geq 2$, then there is a nonconstant holomorphic map from the universal cover of M to complex hyperbolic space.*

We also mention the following result, which follows immediately from 3.2 and the fact that compositions of holomorphic maps are holomorphic.

Theorem 5.5. *If $f: M \to N$ is a holomorphic map and P is a flat G-bundle over N with holomorphic metric, then f^*P is a reductive bundle over M with holomorphic metric.*

This has interesting points of contact with the theory of variations of Hodge structure. In particular, the theorems of Deligne [3] and Griffiths [8] on the complete reducibility of the action of the fundamental group of the base of a variation of Hodge structure on the fiber follow from 5.5 in the case where the base is compact. This follows immediately if the Hodge structures in question are classified by a Hermitian symmetric space. For general Hodge structures, it follows by observing that the composition of a holomorphic map into the classifying space composed with the canonical projection onto a symmetric space is always harmonic. It should be mentioned that M. Nori, in unpublished work, has given a proof of Deligne's theorem in full generality from this perspective.

6. Rigidity of flat bundles over complex hyperbolic manifolds

The purpose of this section is to extract another consequence of 3.4 and 5.3. We shall be dealing with manifolds whose universal cover is the unit ball in C^m, and we shall refer to them as complex hyperbolic manifolds. The main objective will be to prove a rigidity theorem for actions of lattices in $SU(m,1)$ on the unit ball B^n in an n-dimensional complex vector space.

Questions of this sort have already received a fair amount of attention. Most notably, there is the Mostow-Prasad strong rigidity theorem for finite volume complex hyperbolic manifolds. More recently, Goldman and Millson [7] discussed deformations of a cocompact lattice Γ in $SU(m,1)$ as a subgroup of $SU(n,1)$, $n \geq m$. Γ was regarded as a subgroup of $SU(n,1)$ by means of the canonical inclusion of $SU(m,1)$ in $SU(n,1)$. They showed that the action by isometries of Γ on B^n was locally rigid.

Goldman and Millson formulated the following problem. Let ω be the invariant Kähler form (with the proper normalization) on the complex n-ball. By the van Est theorem, it defines an element in the continuous Eilenberg-Mac Lane cohomology group $H^2(PSU(n,1),\mathbf{R})$. Let

$$\omega^m \in H^{2m}(PSU(n,1),\mathbf{R})$$

be the mth exterior power. If $\rho\colon \Gamma \to PSU(n,1)$ is a homomorphism, we obtain a class $\rho^*\omega^m \in H^{2m}(\Gamma,\mathbf{R})$. Suppose Γ is a cocompact, torsion free discrete subgroup of $PSU(m,1)$. Then one can choose a fundamental class $[\Gamma]$ in $H_{2m}(\Gamma,\mathbf{R})$. Define the homological volume $\mathrm{vol}(\rho)$ of ρ to be the absolute value of $\rho^*\omega^m/m!$ evaluated on $[\Gamma]$. Goldman and Millson asked whether the following result was true.

Theorem 6.1. *Suppose M is a compact complex hyperbolic manifold of complex dimension at least two, with fundmental group Γ. If ρ is a homomorphism of Γ into $SU(n,1)$ with $\mathrm{vol}(\rho) = \mathrm{vol}(M)$, then there is a totally geodesic holomorphic embedding of B^m in B^n which is equivariant with respect to ρ.*

Proof. Let $f\colon M \to \Sigma$ be a smooth section of the flat bundle with fiber B^n determined by ρ. Assume first the flat $PSU(n,1)$-bundle on M determined by ρ is reductive. Then by 3.4, we may assume that f is a harmonic section. Since $\mathrm{vol}(\rho)$ is nonzero, f must have real rank $2m$ at some point as a map from the universal cover of M to B^n. By 5.3, f is either holomorphic or antiholomorphic. By choosing the invariant complex structure on B^n appropriately, we may arrange that it is holomorphic.

By Corollary 4.2 on p. 42 of Kobayashi [12], f is distance nonincreasing with respect to the metrics; we are using on M and B^n. We therefore have, in the obvious sense, that $f^*\omega_{B^n} \leq \omega_M$. On the other hand, the equality of volumes implies that the cohomology classes defined by ω_M^m and $f^*\omega_{B^n}^m$ agree up to sign. Since f is holomorphic, they are both positive, and

$$\int_M \omega_M^m = \int_M f^*\omega_{B^n}^m.$$

Given the inequality above, the two forms must be equal, so f is actually an isometric immersion. Now recall the following consequence of the Weitzenböck formula of §4 (and the accompanying notation):

$$\begin{aligned}\Delta|\theta|^2 &= 2\langle\nabla^*\nabla\theta, \theta\rangle - 2|\nabla\theta|^2 \\ &= 2\sum_{i=1}^{2m}\langle\Box^+\theta(e_i), \theta_i\rangle + \sum_{i=1}^{2m}\left\langle\sum_{j=1}^{2m}\theta(\Omega_{ij}^{TM}e_j), \theta_i\right\rangle \\ &\quad + \sum_{i=1}^{2m}\left\langle\sum_{j=1}^{2m}[\Omega_{ji}^+, \theta_j], \theta_i\right\rangle - 2|\nabla\theta|^2 \\ &= 2\sum_{i,j}\langle\theta([[e_j, e_i], e_j]), \theta_i\rangle + 2\sum_{i,j}\langle[[\theta_i, \theta_j], \theta_j], \theta_i\rangle - 2|\nabla\theta|^2.\end{aligned}$$

Since the metric f is an isometric immersion and $e_1, \cdots, e_{2m}$ are an orthonormal basis at each point, the θ_i must all be of unit length and mutually orthogonal. Let $w_1, \cdots, w_{2k}$ be a basis for the space of columns of k complex numbers, regarded as a real vector space. Let w_i^* be the conjugate transpose of w_i. Then define

$$\xi_i = \begin{pmatrix} 0 & w_i \\ w_i^* & 0 \end{pmatrix}.$$

The ξ_i are a basis of $\mathbf{p} \subset su(k, 1)$. A simple calculation shows

$$[\xi_j, [\xi_i, \xi_j]] = \begin{pmatrix} 0 & |w_j|^2 w_i - \langle w_i, w_j\rangle w_j \\ |w_j|^2 w_i^* - \langle w_j, w_i\rangle w_j^* & 0 \end{pmatrix}.$$

Thus, if the w_i were mutually perpendicular and of unit norm, we have

$$[\xi_j, [\xi_i, \xi_j]] = \xi_i, \quad i \neq j.$$

Since this is the case with the e_i and θ_i, we have

$$\theta([[e_j, e_i], e_j]) = \theta_i, \quad i \neq j, \qquad [[\theta_i, \theta_j], \theta_j] = -\theta_i, \quad i \neq j.$$

Furthermore, since θ is an isometric injection at each point, $|\theta|^2$ is constant. Therefore, $|\nabla\theta|^2 = 0$, so f is totally geodesic.

Now we need to justify our assumption that the flat $PSU(n, 1)$-bundle on M is reductive. If it is not, then the image of Γ in $SU(n, 1)$ leaves invariant a nontrivial subspace V of the standard representation in C^{n+1}. Let $V^\perp \subset C^{n+1}$ be the annihilator of V relative to the invariant Hermitian form. $V^\perp$ is an invariant subspace. If the restriction of the invariant Hermitian form to V is nondegenerate, then $V^\perp$ is an invariant complement. Since ρ is not reductive, there must be an invariant subspace such that the restriction of the Hermitian form is degenerate. Let $V' = V \cap V^\perp$. Every element of V' is in the null cone of C^{n+1}. If $v, w \in V'$ are linearly independent, then some linear combination is not contained in the null cone. Thus, V' is of dimension one, and it is invariant, since it is the intersection of invariant subspaces. Hence, $\pi_1 M$ lands in a parabolic subgroup of $PSU(n,1)$ whose inverse image in $SU(n,1)$ leaves invariant a complex line in the null cone of C^{n+1}. We may think of the Lie algebra of $SU(n,1)$ as the set of matrices

$$\left\{\begin{pmatrix} ia & w & b \\ -w^* & S - i(a+c)/(n-1) & v \\ \bar{b} & v^* & ic \end{pmatrix}\right\},$$

where $a, c \in \mathbf{R}$, $b \in C$, $v, w \in C^{n-1}$, and $S \in su(n-1)$. The Lie algebra of the subgroup which leaves the line generated by $(1, 0, \cdots, 0, 1)$ invariant is given by the set of matrices

$$\mathbf{n} = \left\{\begin{pmatrix} i\alpha & w & \beta - i(\gamma+\alpha) \\ -w^* & S + (2i\gamma/(n-1))\mathrm{Id} & w^* \\ \beta + i(\gamma+\alpha) & w & -i(2\gamma+\alpha) \end{pmatrix}\right\},$$

where $S \in su(n-1)$, $w \in C^{n-1}$, and $\alpha, \beta, \gamma \in \mathbf{R}$. The corresponding group is the semidirect product of a nilpotent group and a group of automorphisms of that group. The latter is a product of a compact group and a group of dilatations isomorphic to the multiplicative group in C. The associated graded flat vector bundle has holonomy contained in this group of automorphisms, so its Chern classes vanish. On the other hand, as mentioned in the last section, the Chern classes of this bundle are the same as those of the original one, so we reach a contradiction, unless the original bundle was reductive. q.e.d.

The theorem implies that the subspace of $\mathrm{Hom}(\pi_1 M, PSU(n,1))$ consisting of representations with maximal volume is isomorphic to the space of homomorphisms of $\pi_1 M$ into $U(n-m)/\mu_{n-m}$, where μ_{n-m} is the group of $(n-m)$th roots of unity, since the latter is isomorphic to the centralizer of $PSU(m,1)$ in $PSU(n,1)$. This has been proven by Goldman and Millson for the component containing the standard representation.

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